Proceedings of the American Academy of Arts and Sciences.

Vol. XLVI. No. 20. — April, 1911.

CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE EFFECTS OF SUDDEN CHANGES IN THE INDUCTANCES OF ELECTRIC CIRCUITS AS ILLUSTRATIVE OF THE ABSENCE OF MAGNETIC LAG AND OF THE VON WALTENHOFEN PHENOMENON IN FINELY DIVIDED CORES. CERTAIN MECHANICAL ANALOGIES OF THE ELECTRICAL PROBLEMS.

By B. Osgood Peirce.

With a Plate.

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By B. Osgood Peirce.

Presented May 11, 1910. Received December 31, 1910.

In making some kinds of electrical measurements, one occasionally needs to alter abruptly the inductances of a circuit and to inquire what the effect of the change is upon the march of the currents which the circuit is carrying. If the circuit happens to be a simple one with no magnetic metals and no other circuits near, and if the whole change takes place in a sufficiently short time, it is easy to compute the magnitude and the direction of the corresponding change in the current. If, however, the circuit is complex, or affected by the presence of other inductive circuits in the neighborhood, and if the duration of the change in inductance is long compared with the various time constants which enter, the problem may be much more difficult; though if there be no magnetic metals in the field, the principles laid down more than forty years ago by Maxwell[1] in his dynamical theory of the electromagnetic field, and soon afterwards elaborated and illustrated by Rayleigh and others, point the way to the solution.

In most cases which present themselves in practice, there are masses of magnetizable metal in the form of cores, near the circuits to be studied, and it is often difficult, even if one knows something about the magnetic properties and the history of the cores, to predict exactly what the effects of a given sudden change in the inductances will be.

[1] Maxwell, Philosophical Transactions, Dec. 1864; Rayleigh, Philosophical Magazine, **38**, 1869, **39**, 1870, **30**, 1890.

This paper discusses first, with the help of mechanical analogies, a few simple and familiar cases of circuits without cores, with the purpose of emphasizing some facts to be met with also when cores are present, and then gives a number of diagrams obtained from the photographic records of oscillographs in circuits which contained large electromagnets some with solid and some with divided cores. Changes in the inductances of a circuit which contains one or more electromagnets often involve the moving of comparatively large masses of metal, and it is obviously impossible to make such changes instantaneously even though they may be carried out in intervals which are not long relatively to the time constants of the circuit. In solid cores, also, eddy currents tend to mask the effects of sudden changes in the conformation of the circuit, and this, with the fact that the susceptibility of the iron depends not only upon the intensity of the present excitation, but also upon the past experiences of the metal, leads to considerable differences in the magnetic behavior of a circuit according as it does or does not "contain iron." The diagrams show these differences and illustrate some typical conditions which arise in practical work.

It is well known that the final flux of magnetic induction through a solid iron core is not determined by the intensity of the excitation alone even when its magnetic condition at the outset is given, but depends in many cases upon the manner of application of the given excitation, — whether it be made suddenly, by small steps at intervals, or by slow, continuous rise. The diagrams are interesting in this connection because they show that when the core is fairly well divided, the forms of two distinct portions of a current curve, interrupted by a sudden change of inductances, are often almost identical with corresponding portions of two current curves obtained without any such interruption, the one with the original inductances, the other with the final ones.

It may be well to consider briefly at the outset the very simplest case, the familiar one of a single circuit, without iron, of fixed resistance, r ohms, and of inductance originally equal to L_0 henries, which contains a constant electromotive force of E volts and is carrying at the time $t = 0$, a current of C_0 amperes. At this instant ($t = 0$) let the inductance begin to change according to some law, and progressing always in the same direction, let it attain at the time T the given value L_1, after which it shall remain constant.

In order to illustrate graphically the effect of making the given change in inductance in longer or shorter time intervals, it will be convenient to use three rectangular axes for t, C, and T respectively,

and to represent the value (L) of the inductance at any time $0 < t < T$, by the expression

$$L_0 + (L_1 - L_0) f\left(\frac{t}{T}\right), \tag{1}$$

where $f(0) = 0, f(1) = 1$, and for $0 < x < 1, f'(x) > 0$.

The line OK in the tT plane (Figure 1) has the equation $t = T$, and after the value ($T' = OF$) has been fixed for T, the course of the current during the change of inductance may be shown by a curve (HQS) in a plane (FRS), the equation of which is $T = T'$. FH, which measures the ordinates of the line AG in the CT plane represents the initial current (C_0) at the time $t = 0$, and the ordinates of the curve MB in the plane COK represent for different values of T the intensities of the current at the end of the change in inductance, when the form of the function f is the same. The vertical distance AB shows the magnitude of the sudden change in the current strength when the change in L is supposed to be instantaneous and is the same whatever form f has.

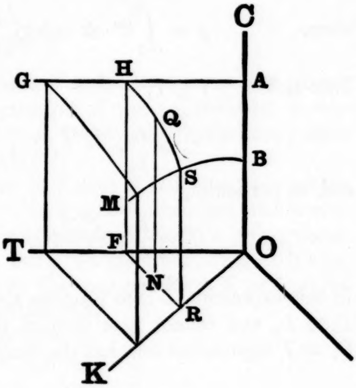

FIGURE 1.

During the interval $0 < t < T$, the current is to be found, of course, by solving the equation

$$E - \frac{d\,(LC)}{dt} = rC, \tag{2}$$

— where L has the variable value given above — and making the constant of integration such that $C = C_0$ when $t = 0$. After the epoch $t = T$, the intensity of the current satisfies the equation

$$E - L_1 \cdot \frac{dC}{dt} = rC, \tag{3}$$

in which the coefficients are constants. Equation (2) may be written in the form

$$\frac{dC}{dt} + \frac{r + L'}{L} \cdot C = \frac{E}{L}, \tag{4}$$

where $L' = dL/dt$, and if we make use of the usual notation,[2] and put $P = (r + L')/L$, $Q = E/L$, we shall have in general

$$C = e^{-p}(M + \int_0^t Qe^p dt), \tag{5}$$

where $$p = \int_0^t P \cdot dt = \log\left(\frac{L}{L_0}\right) + \int_0^t \frac{rdt}{L}. \tag{6}$$

That is, if $0 \leqq t \leqq T$,

$$C_t = \frac{L_0}{L_t} e^{-\int_0^t \frac{rdt}{L}} (C_0 + \frac{E}{L_0} \int_0^t e^{\int_0^t \frac{rdt}{L}} \cdot dt), \tag{7}$$

and, in particular,

$$C_T = \frac{L_0}{L_1} e^{-\int_0^T \frac{rdt}{L}} (C_0 + \frac{E}{L_0} \int_0^T e^{\int_0^t \frac{rdt}{L}} \cdot dt). \tag{8}$$

In this expression L is to progress always in the same direction from L_0 to L_1, and cannot pass through the value zero, so that the limit of C_T as T approaches zero has the familiar value

$$\operatorname*{Limit}_{T \doteq 0} C_T = \frac{L_0 C_0}{L_1}, \tag{9}$$

which might have been found directly by integrating (2) with respect to t from 0 to T; the electromagnetic momentum has no sudden change. Equation (9) follows immediately, of course, when one makes use of the usual analogies between the phenomena of ordinary mechanics and those of electromagnetism. Equation (2) is in form like the equation of motion of a system the mass of which changes with the time in a certain given manner and which is under the action of a constant accelerating force and a retarding force proportional to the velocity. Let a moving mass L grow steadily during its motion by the gradual accretion of small particles which, originally at rest, are suddenly made part of the moving system, much as the links of a fine chain which has been lying on a table are successively set in motion when one end of the chain is lifted more and more; or let the mass L decrease steadily by the loss of small particles each of which leaves

[2] Forsyth, Treatise on Differential Equations, § 14.

the system with a parting push which reduces its own velocity to zero and speeds its late companions on their way; then, if C is the velocity of the moving mass, E the accelerating force, and rC the retarding force, the equation of motion will be $d\,(LC)/dt = E - rC$, that is, (2).

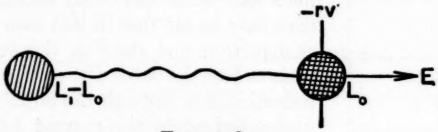

FIGURE 2.

If, when its velocity is C_0, the mass of such a system be instantaneously changed from L_0 to L_1, the principle of the conservation of momentum in impact shows that if C_1 is the velocity immediately after the impulsive change, $L_0C_0 = L_1C_1$.

The conventional diagram shown in Figure 2 indicates the nature of this simple mechanical problem. L_0 is a mass furnished with a stiff vane of such a size as to make the air resistance (which is proportional to the velocity) equal to r units when the mass is moving with unit velocity. L_0 is urged to the right by the constant force E and is retarded by a force rv. A slack inextensible string connects L_0 with another mass $L_1 - L_0$, and when the string becomes taut, the impulsive change in the velocity of L_0 corresponds to the change in the current in the inductive circuit when the inductance is impulsively changed from L_0 to L_1.

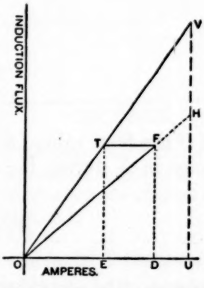

FIGURE 3. The line OFTV represents the change of the induction flux linked with a circuit without iron, when the inductance is suddenly increased.

If the induction flux, N, in a circuit which contains no iron be plotted against the current, the resulting locus is a straight line through the origin, the slope of which is the self-inductance of the circuit. If, then, the lines (OH, OV, Figure 3) corresponding to L_0 and L_1 be drawn, and if when the rising current has attained the value C_0, the inductance be supposed to change suddenly to L_1, the induction flux through the circuit preserves its value unchanged while the current falls from C_0 to C_1, and the point in the diagram which gives the state of the circuit moves from F to T.

If, as is approximately the case with some circuits which have open

cores made of very finely divided soft iron, the hysteresis diagram is extremely narrow, so that the inductance may be considered to be a definite function of the current strength, we may represent two different states of the circuit by lines like OFR and OKS of Figure 4. It is then easy to see that in this case also a sudden change from one state to the other when the current had the value OD $= C_0$ would leave the induction flux through the circuit momentarily unchanged while the current fell to OE $= C_1$, and the point which represents the state of the circuit would suddenly move from F to K.

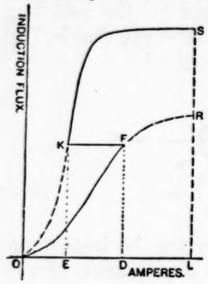

FIGURE 4. The line OFKS represents the growth of the induction flux in a magnet with finely divided core when the inductance of the circuit is suddenly increased.

If, at any instant, the total flux of magnetic induction through any simple circuit, which may or may not contain iron, is N (maxwells), if r is the resistance of the circuit in ohms, C, the current in amperes, and E, the applied electromotive force in volts,

$$E - \frac{1}{10^8} \cdot \frac{dN}{dt} = rC, \tag{10}$$

or

$$\frac{dN}{dt} = 10^8 \cdot r \left(\frac{E}{r} - C\right), \tag{11}$$

and if the final value (E/r) of the current be denoted by C', and the change in N during the time interval t_1 to t_2 by $N_{1,2}$,

$$N_{1,2} = r \cdot 10^8 \int_{t_1}^{t_2} (C' - C)\, dt. \tag{12}$$

If, now, C be plotted against the time (as in an oscillograph diagram) in a curve s (Figure 5) in which l centimeters parallel to the axis of abscissas represent one second, and an ordinate m centimeters long one ampere, the curve will have a horizontal asymptote (CY) at a distance (KC) corresponding to E/r amperes from the time axis, and, if OK represents the time t_1 and OL the time t_2, the area FGDC, or $A_{1,2}$, expressed in square centimeters, is equal to

$$lm \int_{t_1}^{t_2} (C' - C)\, dt, \tag{13}$$

and $$N_{1,2} = \frac{10^8}{lm}(r \cdot A_{1,2}). \tag{14}$$

The curve ONJ of Figure 6, which has been carefully drawn to scale, represents the growth of the current with the time in a circuit without

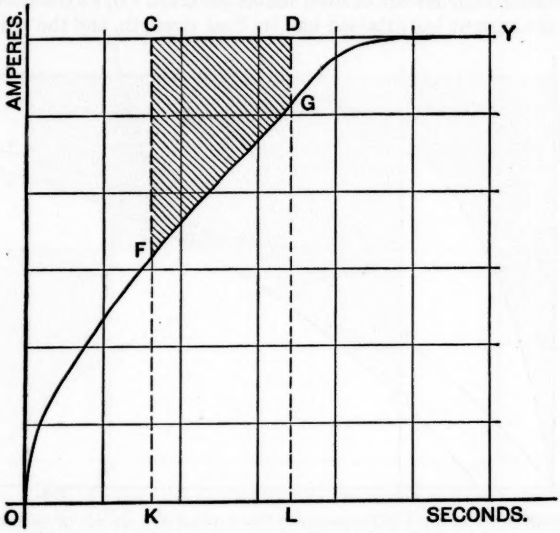

FIGURE 5. If l centimeters parallel to the horizontal axis represent one second, and an ordinate m centimeters long one ampere, $A \cdot 10^8 \cdot r/lm$ (where A is the area, in square centimeters, of CDGF) represents the change in the magnetic flux through the circuit during the interval KL.

iron, of resistance r and inductance L. The curve OPT represents the current in the same circuit when the inductance has been increased to $4L$, while the resistance is the same as before. If, when the inductance of the circuit is L, the current rises in the time OU to the value UN, and if then the inductance is instantly increased to $4L$, the current falls to UF and then rises again in the manner indicated by the curve FG, which is the curve OPT moved to the right through a distance OL just great enough to make its ordinate at the time OU equal to one fourth of UN. Since the area between the curve and its asymptote is proportional to the inductance flux through the circuit, it is clear without any of the reasoning of the preceding paragraphs, that

there cannot be any impulsive change of the induction flux when the inductance is suddenly increased. A glance at the figure shows, however, that the rate of increase of the induction suddenly becomes much greater than it was just before the change.

Curve ODE of Figure 7 shows the manner of growth of the current in another simple circuit of fixed inductance, $4L$. If, at the time OW, when the current has attained half its final strength, and the induction

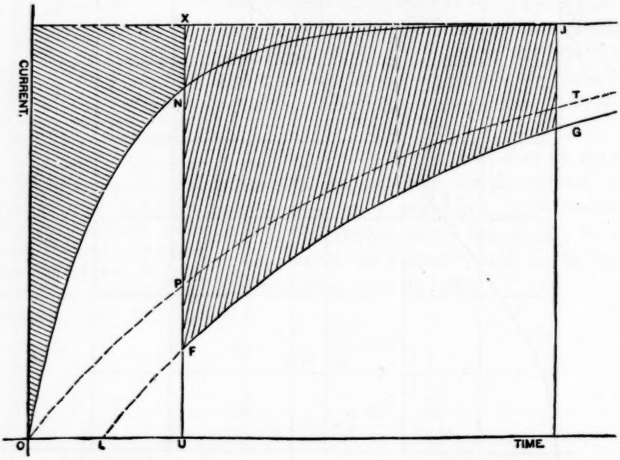

FIGURE 6. The line ONJ represents the current in a circuit of inductance L without iron. OPT shows the form of the current in the same circuit when the inductance has been increased to $4L$. ONFG is the current when the inductance is suddenly changed from L to $4L$ at the time OU.

flux through the circuit is represented on the scale indicated by equation by OABD, the inductance be suddenly changed to L, the current suddenly becomes four times as strong as it was and then falls in a manner shown by the curve ST. The flux through the circuit just after the change is already twice as large as it will be eventually when the current reaches its final value, OA, and it decreases by an amount represented by the area BST, which is half the area AODB. Just before the change the flux was increasing with the time at a rate represented by the length of the line BD; just after the change it decreases at a rate represented by the line BS, which is twice as long as DB.

In the case of a circuit which does not contain iron, an increase of

inductance without an increase of the resistance usually involves a change of the conformation of the circuit, and this generally requires a considerable fraction of a second, at least, to bring about, so that the formula (9) cannot be used to determine the current strength at the end of the inductance change. To illustrate this fact we may assume that the change from L_0 to L_1 in the time T is brought about at a

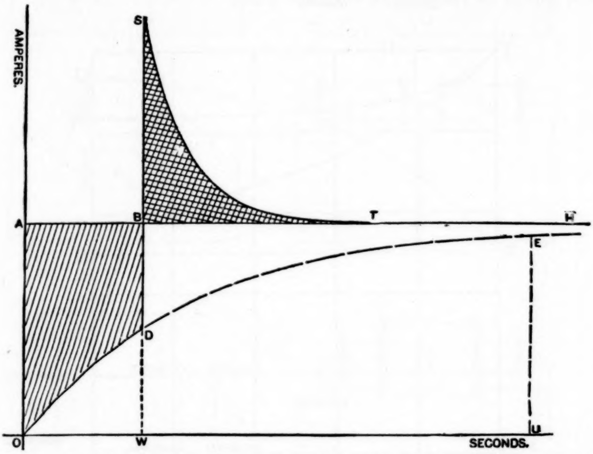

FIGURE 7. ODE shows the current in a simple circuit of fixed inductance, $4L$. If at the time OW, when the current has attained half its final intensity, the inductance is suddenly reduced to L, the course of the current will be ODBST.

constant rate so that $L = L_0 + t\,(L_1 - L_0)/T$, and the strength of the current at the time t is given by the equation

$$C = \left(\frac{L_0}{L}\right)^m \cdot \left(C_0 + \frac{ET}{m\,(L_1 - L_0)} \cdot \frac{L^m - L_0{}^m}{L_0{}^m}\right), \qquad (15)$$

where

$$m = \frac{rT + L_1 - L_0}{L_1 - L_0}. \qquad (16)$$

If, now, $C = E/r$ amperes, $L_0 = 2$ henries, $L_1 = 4$ henries, then, according as T is one second, half a second, one tenth of a second, or one hundredth of a second, the value (C_1) of the current at the end of the interval T is $0.980 \cdot C_0$, $0.962 \cdot C_0$, $0.836 \cdot C_0$, or $0.569 \cdot C_0$, whereas C_1

would be $0.5 \cdot C_0$, if the change in the inductance had been instantaneous.

Figure 8 shows in TW the relative changes in the current in this circuit from $t = 0$ to $t = T$, when T is one tenth of a second, and in TZ the changes when T is one one-hundredth of a second. If the change were instantaneous the course of the current in one tenth of a second would correspond to the line TRU.

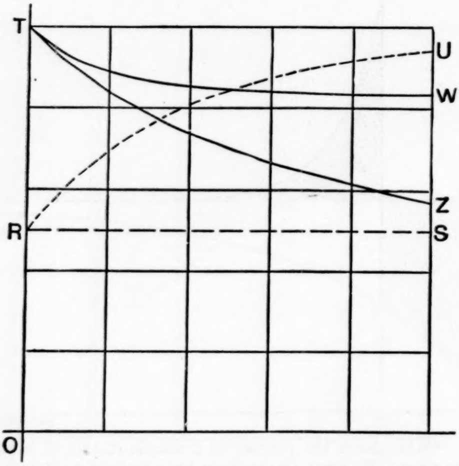

FIGURE 8. If in a certain inductive circuit, without iron, the inductance be instantaneously doubled, the course of the current in the next tenth of a second will be TRU. TW shows the current if the doubling be brought about by a continuous change going on at a constant rate during the whole interval. TZ shows on a different time scale, the course of the current for a hundredth of a second, if during this interval the inductance be changed at a constant rate which results at the end in its being doubled.

We may next consider the somewhat less simple circuit indicated in Figure 9_a, consisting of three parallel branches each of which has self-inductance, but no two of which have mutual inductance. Let r, r_1, r_2 be the resistances of the branches, L, L_1, L_2 their inductances, E, E_1, E_2 the constant electromotive forces of the generators in them, and C, C_1, C_2 the currents. At the time $t = 0$, when the currents and the inductances have given values, let the inductances begin to change according to given laws each of which can be expressed by an equation similar to (1), and let them attain, at the time T, other given values, which they

thereafter keep. It is evident that any instant during the interval $0 < t < T$,

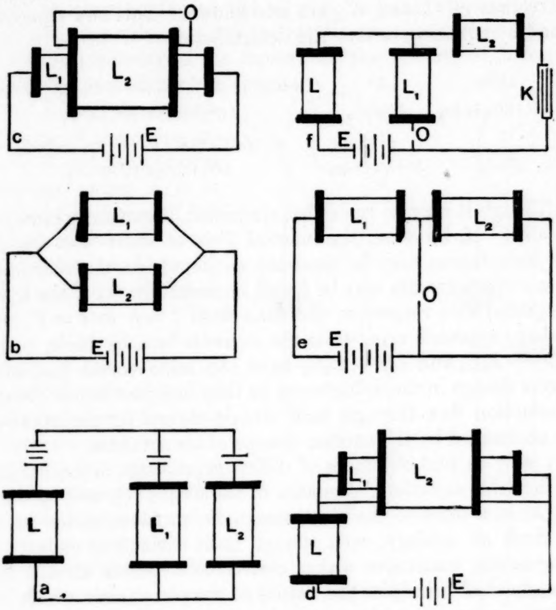

FIGURE 9.

$$E + E_1 - \frac{d\,(LC)}{dt} - \frac{d\,(L_1C_1)}{dt} = rC + r_1C_1,$$

$$E + E_2 - \frac{d\,(LC)}{dt} - \frac{d\,(L_2C_2)}{dt} = rC + r_2C_2, \qquad (17)$$

$$C = C_1 + C_2,$$

or, if we represent differentiation with respect to the time by accents,

$$(L + L_1)\,C'_1 + LC'_2 + (L' + L'_1 + r + r_1)\,C_1 + (L' + r)\,C_2 = E + E_1,$$

$$LC'_1 + (L + L_2)\,C'_2 + (L' + r)\,C_1 + (L' + L'_2 + r + r_2)\,C_2 = E + E_2. \qquad (18)$$

If, from these equations and others obtained by differentiating them with respect to the time, C_2 and its derivatives be eliminated, we shall

get a differential equation of the second order for C_1 in which the inductances and their derivatives are known functions of t, and the initial values of C_1 and C'_1 are also known. This new equation may be found by equating to zero the determinant

$$\begin{vmatrix} L & 2L'+r & L'' & (L+L_1)C''_1+(2L'+2L'_1+r+r_1)C'_1+(L''+L''_1)C_1 \\ L+L_2 & 2L'+2L'_2+r+r_2 & L''+L''_2 & LC''_1+(2L'+r)C'_1+L''C_1 \\ 0 & L & L'+r & (L+L_1)C'_1+(L'+L'_1+r+r_1)C_1-E-E_1 \\ 0 & L+L_2 & L'+L'_2+r+r_2 & LC'_1+(L'+r)C_1-E-E_2 \end{vmatrix} \tag{19}$$

and, although it may be somewhat simplified, it generally proves rather intractable. If, however, the interval T is so short that the changes in the inductances may be regarded as impulsive, the corresponding changes in the currents may be found immediately, for if the equations be integrated with respect to the time from $t = 0$ to $t = T$, and if T be made to approach zero, while the currents remain finite, it appears that $LC + L_1C_1$ and $LC + L_2C_2$ have the same values just after the impulsive change in the inductances as they had just before the change. The induction flux through each circuit chosen for the equations remains unchanged by the sudden change of inductances.

It is easy to find a number of different problems in mechanics each of which yields equations of motion of the form (17), and is, therefore, analogous in a sense to the electromagnetic problem under consideration. Such an analogy, even though it be difficult to embody it in a working model, sometimes makes clearer to a person already familiar with mechanical principles the nature of the phenomena which he is to look for in interpreting his electrical equations. It will do no harm if, in imagining a mechanical system which is to serve this purpose, we postulate the existence of flexible, inextensible, massless strings, or even, at a pinch, the existence of stiff, nearly massless rods, or of pulley wheels so light that their moments of inertia shall be negligible. It is often desirable to imagine the motions of the masses which in the mechanical system represent the inductances in the electrical problem, to be hindered by retarding forces proportional to the velocities, to represent the electrical resistances. The resistance which the air offers to a body moving through it with a constant velocity not greater than 50 cms. per second is very nearly proportional to that velocity; and since the velocities which in the mechanical case correspond to the currents are usually much smaller than that, the resistance may be sufficiently well indicated by thin wings or vanes of proper size attached to the masses.

In the arrangement shown in Figure 10 the masses L, L_1, L_2, are

urged towards the bottom of the diagram by forces of intensity E, E_1, E_2. The lines drawn across the masses indicate wings of such shapes as to make the resistances due to the air r, r_1, r_2 dynes respectively, when the corresponding velocities are one centimeter per second. It is evident from the geometry of the figure that the velocity of L downward is equal to the sum of the velocities of L_1 and L_2 upward. The tension of the string attached to L and passing over the massless pulley A is at every instant half that of the cord which is attached to the massless pulley B, and equal to the tension of the cord which connects L_1 and L_2. The equations of motion of the masses are of the form (17). If, as a consequence of applied forces or impulses, the string should become slack, the analogy between the mechanical and the electromagnetic problems would disappear, and it is sometimes convenient to imagine the masses attached to taut endless strings in some such manner as is shown in Figure 11. It is very easy to construct a model of this kind which will work fairly well if one uses for masses properly loaded roller skates which move about on the level top of a table. The masses may be connected by fine catgut passing around small, cheap pulleys with vertical axes mounted on the table.

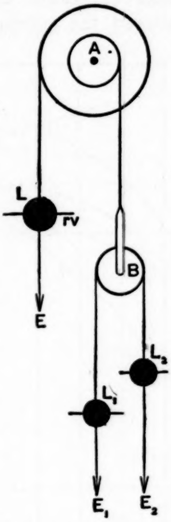

FIGURE 10.

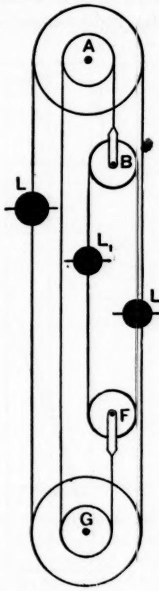

FIGURE 11.

A special case of some practical interest is that indicated in Figure 9_b, where the terminals of a battery without sensible self-inductance are connected by two inductive branches in parallel. The currents are given by the equations

$$L_1 L_2 \frac{d^2 C_1}{dt^2} + [(r + r_1) L_2 + (r + r_2) L_1] \frac{dC_1}{dt} + (r_1 r_2 + r r_1 + r r_2) C_1 = r_2 E \quad (20)$$

$$L_1 L_2 \frac{d^2 C_2}{dt^2} + [(r + r_1) L_2 + (r + r_2) L_1] \frac{dC_2}{dt} + (r_1 r_2 + r r_1 + r r_2) C_2 = r_1 E,$$

and it is clear that if the inductances are suddenly changed, the products $L_1 C_1$ and $L_2 C_2$ are continuous, and if, in particular, only one of the inductances is altered, the current in the parallel branch is itself

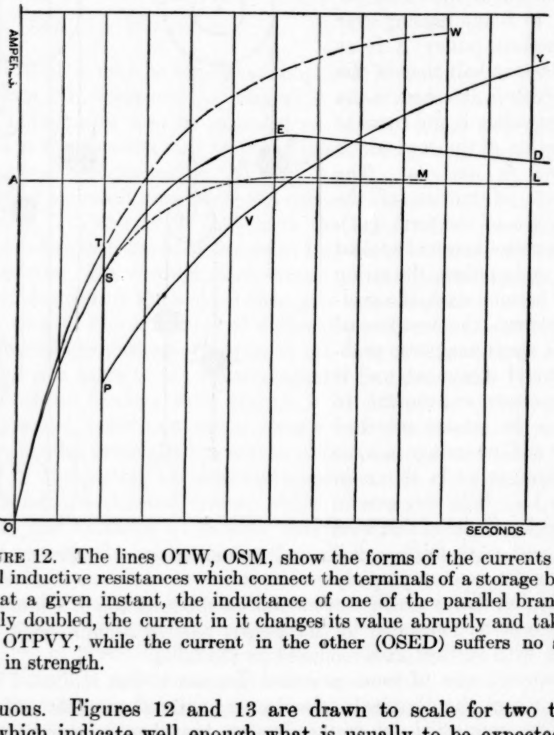

FIGURE 12. The lines OTW, OSM, show the forms of the currents in two parallel inductive resistances which connect the terminals of a storage battery. When at a given instant, the inductance of one of the parallel branches is suddenly doubled, the current in it changes its value abruptly and takes the course OTPVY, while the current in the other (OSED) suffers no sudden change in strength.

continuous. Figures 12 and 13 are drawn to scale for two typical cases which indicate well enough what is usually to be expected. In both diagrams $L_1 = 1$, $L_2 = 1$, $r = 12$, $r_1 = 20$, $r_2 = 30$, $E = 120$ (or these quantities are to be in the proportions here given). The final values of C_1 and C_2 are 3 amperes and 2 amperes.

In the case which corresponds to Figure 12 the battery circuit is closed at a given instant, and 0.02 seconds afterwards, when C_1 has

attained the value 1.607 and C_2 the value 1.457, L_1 is suddenly changed from 1 to 2. As a consequence, C_1 falls suddenly to 0.8035, while C_2 remains momentarily unchanged. Before the change, the currents were given by the equations

$$C_1 = 3 - \tfrac{120}{104}\, e^{-24t} - \tfrac{120}{65}\, e^{-50t}, \quad C_2 = 2 + \tfrac{120}{156}\, e^{-24t} - \tfrac{360}{130}\, e^{-50t}, \quad (21)$$

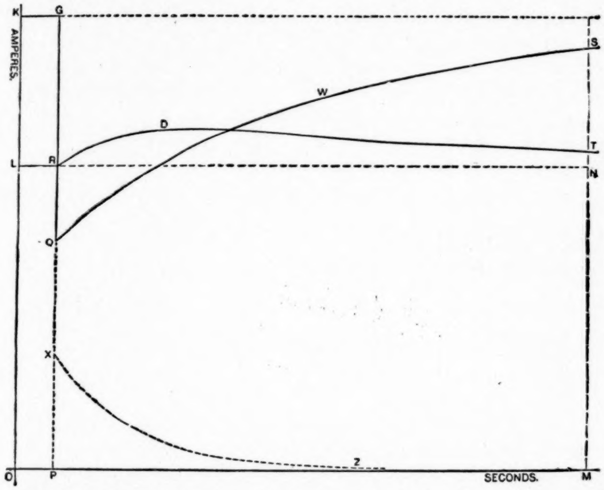

FIGURE 13. After the currents in two parallel inductive resistances which connect the terminals of a storage battery have become steady, at the values OK, OL, the inductance of one of the branches is suddenly doubled so that the current in it takes the course KGQWS. The current in the other branch takes the continuous form LRDT and approaches its final value from above.

and afterwards by the approximate equations

$$\begin{aligned} C_1 &= 3 - 1.912\, e^{-13.48t} - 0.284\, e^{-44.52t}, \\ C_2 &= 2 + 0.803\, e^{-13.48t} - 1.346\, e^{-44.52t}. \end{aligned} \quad (22)$$

The line OTPVY shows the course of C_1, and OSED the course of C_2. It will be observed that C_2 approaches its final value from above.

If the change in inductance is made after the currents have attained their final values, the courses of C_1 and C_2 will be those indicated in Figure 13 by the lines KGQS and LRDT. If after the currents have reached their steady values, the main circuit be suddenly broken, C_1

and C_2 instantly acquire equal and opposite values, and the subsequent course of C_1 is given by the equation

$$C_1 = \frac{E(L_1r_2 - L_2r_1)}{r_1r_2 + rr_1 + rr_2} e^{-\frac{r_1+r_2}{L_1+L_2}t}. \tag{23}$$

See in Figure 13 the line GRXZ.

If the terminals of an open battery circuit of inductance L and of resistance r be connected by a number of inductive conductors in parallel, of resistances r_1, r_2, r_3, r_4, etc., and of inductances L_1, L_2, L_3, L_4, etc., and if sudden changes be made in the inductances, the quantities

$$LC + L_1C_1,\ LC + L_2C_2,$$
$$LC + L_3C_3,$$

etc., will be continuous. If L is negligible, and if only some of the other inductances be impulsively changed, the currents in the other branches will be continuous.

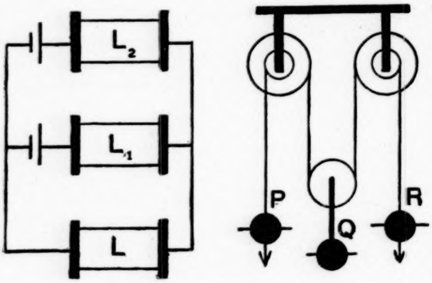

FIGURE 14.

If, in the arrangement shown in Figure 14, the masses P, Q, R are numerically equal to L_1, L, L_2, respectively, if the velocities of P and R in the direction of the bottom of the page are C_1, C_2, and if the dimensions of the vanes attached to the masses are such that the air offers resistance of r_1, r, r_2 times the velocities to the motion of P, Q, and R, the equations of motion of the masses are identical with the current equations for the electrical circuit shown in the figure.

The currents in two neighboring circuits (Figure 9_c) of self-inductances L_1, L_2, and mutual inductance M, which contain the electromotive forces E_1, E_2, are given by the familiar equations

$$\left.\begin{aligned} E_1 - L_1\frac{dC_1}{dt} - M\frac{dC_2}{dt} - r_1C_1 = 0, \\ E_2 - L_2\frac{dC_2}{dt} - M\frac{dC_1}{dt} - r_2C_2 = 0, \end{aligned}\right\} \tag{24}$$

and any impulsive changes in the inductances cause such sudden changes in the current as will keep $L_1C_1 + MC_2$ and $L_2C_2 + MC_1$ momentarily unchanged.

Many different working models have been made to illustrate the simple electrical problems which concern two such circuits. Of these some of the best known are due to Maxwell, Rayleigh, J. J. Thomson, Webster, and Boltzmann.

The original model of Maxwell, now in the Cavendish Laboratory, is represented by Figure 15_a, taken from Gray's Absolute Measurements in Electricity and Magnetism, where an excellent account of the apparatus and its theory may be found.

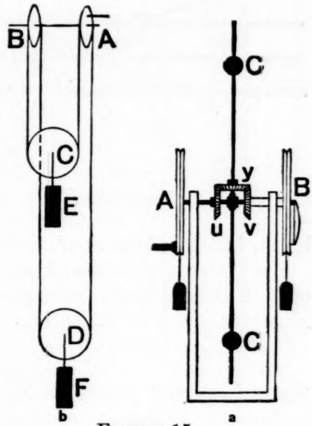

FIGURE 15.

In Lord Rayleigh's model, shown in Figure 15_b, "two similar pulleys A, B, turn upon a piece of round steel fixed horizontally. Over these is hung an endless cord, and the two bights carry similar pendent pulleys, C, D, from which again hang weights, E, F. . . . In the electrical analogy, the rotary velocity of A corresponds to a current in a primary circuit, that of B to a current in the secondary. . . . In the absence of friction there is nothing to correspond to electrical resistance, so that the conductors must be looked on as perfect. If x and y denote the circumferential velocities, in the same direction, of the pulleys A, B, where the cord is in contact with them, $\frac{1}{2}(x + y)$ is the vertical velocity of the pendent pulleys. Also $\frac{1}{2}(x - y)$ is the circumferential velocity of C, D, due to rotation, at the place where the cord engages. If the diameter be here $2a$, the angular velocity is $(x - y)/2a$. Thus, if M be the total mass of each pendent pulley and attachment, Mk^2, the moment of inertia of the revolving parts, the whole kinetic energy corresponding to each is

$$\frac{1}{2} M \left\{ \frac{(x+y)^2}{4} + \frac{k}{a^2}\left(\frac{(x-y)^2}{4}\right) \right\}. \tag{25}$$

For the energy of the whole system, we should have the double of this, and, if it were necessary to include them, terms proportional to x^2 and y^2, to represent the energy of the fixed pulleys."

Here $L_1 = L_2 = a^2 + k^2$, $M = a^2 - k^2$, and, if there were no magnetic leakage, k would need to be zero.

Figure 16 represents the model of Professor Sir J. J. Thomson.[3] "It consists of three smooth, parallel, horizontal steel bars on which masses m_1, m_2, m slide, the masses being separated from the bars by friction wheels; the three masses are connected together by a light rigid bar which passes through holes in swivels fixed on the upper part of the masses; the bar can slide backwards and forwards through these holes, so that the only constraint imposed by the bar is to keep the masses in a straight line."

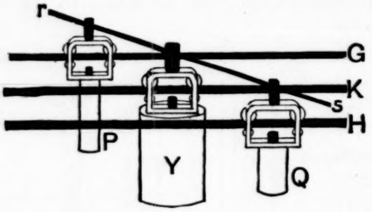

FIGURE 16.

If x'_1, x'_2 are velocities of m_1, m_2, in the same direction, the velocity of M, if it be midway between m_1 and m_2, is $\frac{1}{2}(x'_1 + x'_2)$, and the kinetic energy is of the form

$$\tfrac{1}{2} L_1 x_1'^2 + M x'_1 \cdot x'_2 + \tfrac{1}{2} L_2 x_2'^2, \tag{26}$$

where $L_1 = m_1 + \frac{1}{4} m$, $L_2 = m_2 + \frac{1}{4} m$, $M = \frac{1}{4} m$.

Professor Webster's model is a modification of that of Thomson. "If the middle weight, instead of rolling on a fixed rail, roll on the bar connecting the two other carriages, the coefficients of induction will vary with the position of the middle mass, and moving it along its bar while one of the outer masses is moving will cause the other to move. The centrifugal force tending to make the middle mass roll along its bar will represent the magnetic forces between the currents."

The very elaborate and ingenious model of Boltzmann is described at length in the first fifty pages of his Vorlesungen über Maxwell's Theorie der Elektricität und des Lichtes.

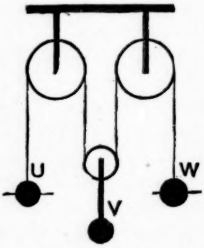

FIGURE 17.

The general features of another simple model illustrative of this electrical problem are shown in Figure 17. The mass of U is $L_1 - M$, that of V is $4M$, and that of W, $L_2 - M$. In Figure 18 the strings

[3] J. J. Thomson, Elements of the Mathematical Theory of Electricity and Magnetism, Chapter XI.; Webster, Science, Dec. 1895; The Theory of Electricity and Magnetism, § 71.

are represented as stretched over four small pulleys to keep them taut. I have found that this model made of three weighted roller skates, moving over a level table top, and connected in the manner indicated by cords passing around such small cheap pulleys as are obtainable at any ironmonger's shop, may be made to work extremely well. The effects of sudden changes of inductance can be directly observed by dropping suitable masses into the skates as they move. In Figure 14, which illustrates the same problem, the mass of Q is M, and those of P and Q are $L_1 - M$, $L_2 - M$, respectively. Q should have no vane.

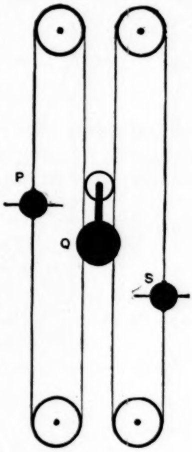

FIGURE 18.

Scores of other models, more or less simple of construction, can easily be devised. It is to be noticed, however, that in some of the models which have been used to illustrate this problem, the masses representative of some of the combinations of the inductances would need to be negative if they were to correspond to cases which occasionally arise in electrical engineering.

If either of the two neighboring circuits contains no battery, the corresponding value of E in equations (24) is to be put equal to zero. Figure 19 is drawn for the case of an induction coil without iron and with no cell in the secondary circuit. The self-inductances of the two circuits are equal. The dotted curve, P, shows the form of the current induced in the secondary circuit when the primary circuit, which has been carrying a steady current, is suddenly broken. If, after a few seconds, the primary circuit containing its battery be closed again, the current in the secondary circuit will have the general form of either Q or S. Q, R, and S are drawn for mutual inductances respectively half as great, nine tenths as great, and equal to, the inductance of either circuit. P is drawn for $M = L/2$, and corresponds, therefore, to Q; the areas V and W are equal. Curves like P corresponding to R and S could be found by exaggerating all of P's ordinates in the ratio 9/5, or the ratio 2.

Figures 20, 21, 22, 23, and 24 illustrate some phenomena which are frequently encountered in the practical use of neighboring inductive circuits. The curves have been drawn to scale for certain numerical values of the resistances, and the inductances so chosen as to make the results typical. There is no iron in either circuit, and only one circuit, the primary, contains a battery.

In Figure 20 the current in the primary circuit is drawn above OX, in the curve OJAZ, and the current in the secondary circuit beneath MN. Each of the circuits has a self-inductance of 2 henries, and the

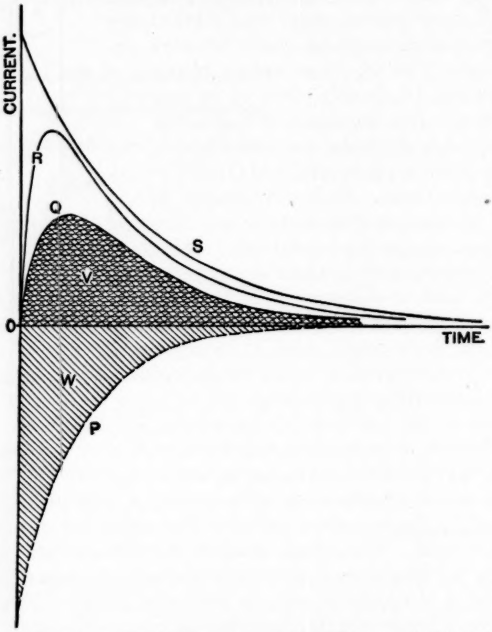

FIGURE 19. The curves Q, R, S represent for different relative values of the mutual inductance the current induced in the secondary circuit of a certain induction coil without iron, when the primary circuit is suddenly closed.

resistances are 30 ohms and 40 ohms. The mutual inductance is at first $\sqrt{2}$, and the currents are given by the equations

$$C_1 = 4 - 2.4\,e^{-10t} - 1.6\,e^{-60t}, \quad C_2 = \frac{-2.4}{\sqrt{2}}\,(e^{-10t} - e^{-60t}), \quad (27)$$

until the time $OG = 1/20$, when all the inductances are suddenly doubled. The currents are then given by the equations

$$C'_1 = 4 - 1.928\,e^{-5t} - 0.840\,e^{-30t}, \quad C'_2 = 0.891\,e^{-30t} - 1.361\,e^{-5t}. \quad (28)$$

In the case represented by Figure 21, $r_1 = 30$, $r_2 = 40$, $M = \sqrt{2}$, $L_2 = 2$. At the beginning $L_1 = 2$, but at the time OA this is suddenly changed to 4. Before the change in L_1 the currents are given by the equations

$$C_1 = 4 - 2.4\, e^{-10t} - 1.6\, e^{-60t}, \qquad C_2 = \frac{-2.4}{\sqrt{2}}\,(e^{-10t} - e^{-60t}), \qquad (29)$$

Just before the impulse, $C_1 = 2.465$, and $C_2 = -0.945$; just after, the current in the primary is about 0.822 and the secondary current has the small positive value 0.217. The new currents satisfy the equations

$$C'_1 = 4 - 2.634\, e^{-20t/3} - 0.543\, e^{-30t}, \; C'_2 = -0.932\, e^{-20t/3} + 1.149\, e^{-30t}, \quad (30)$$

very nearly. C_2 is plotted below TQ.

Figure 22 shows the manner of growth of two neighboring currents, when $r = 30$, $r = 40$, $L = 2$, $L = 2$, and when M, which is at first

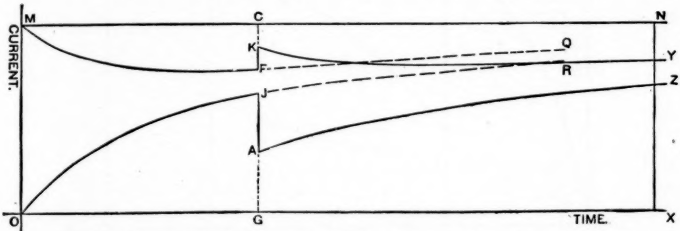

FIGURE 20. OJR and MFQ represent the forms of the primary and secondary currents in a certain induction coil without iron when the primary circuit is closed at the origin of time. If at the time OG, the self-inductances of both circuits and the mutual inductance of the two are suddenly doubled, the currents take the forms OJAZ and MFKY.

$\sqrt{2}$, is suddenly changed to zero at the time OA. When M is changed, the current in the primary circuit suddenly falls from 2.465 to 1.797, and the current in the secondary circuit, which has been negative, rises from -0.945 to $+0.798$. After the change, the currents are given by the simple equations

$$C'_1 = 4 - 2.203\, e^{-15t}, \qquad C'_2 = 0.798\, e^{-20t}. \qquad (31)$$

Figure 23 exhibits the effects of a sudden change in the value of the mutual inductance between the two circuits already described under Figures 21 and 22, from $\sqrt{2}$ to 1.9, while the other inductances remain unaltered. The primary current is shown by the curve OKRGS, and

OX shows its final value. The second current is represented by the curve ADWU plotted under AN and displaced to the right so that the sudden increase in the absolute strength at the time of the change in M may appear. The flux of magnetic induction through the primary circuit is represented on the usual scale by the shaded area. The

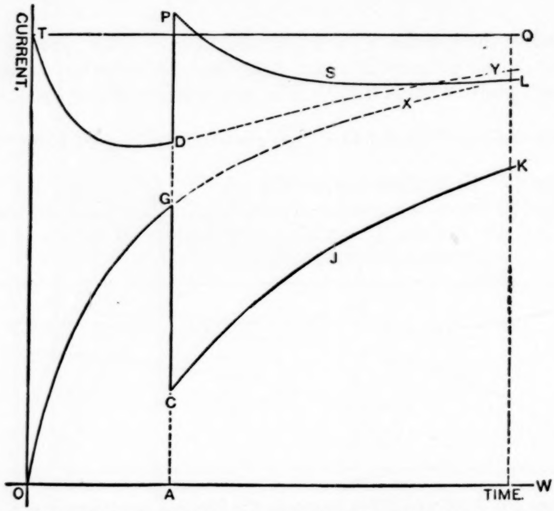

FIGURE 21. At the time OA, the self-inductance of the primary circuit of a certain induction coil without iron is suddenly doubled, while the self-inductance of the secondary circuit and the mutual inductance of the two remain unchanged. OGCJK shows the course of the primary current and TDPSL that of the secondary current.

black area points to a decrease in this flux which goes on from the time XF to the time XG, when the current falls below its final value. The induction flux linked with each of the two circuits is plotted against the time in Figure 24. These quantities are shown to be continuous at the time of change in the inductance, as, of course, they should be.

Figure 25 shows another arrangement of two neighboring circuits and an analogous mechanical system. The gap O is closed at first, but is suddenly opened when the current has become steady. The mass W moves alone under the action of a force E which urges it in the direction of the bottom of the page, and the air resistance. The motion

soon becomes steady, but when the string which connects W to X becomes taut the motion is suddenly changed.

Figure 9_d represents a circuit consisting of three parallel branches, each of which has self-inductance and may contain a battery, and two of which have mutual inductance. If L, L_1, L_2 are the self-inductances, r, r_1, r_2 the resistances, C, C_1, C_2 the currents, E, E_1, E_2 the electro-

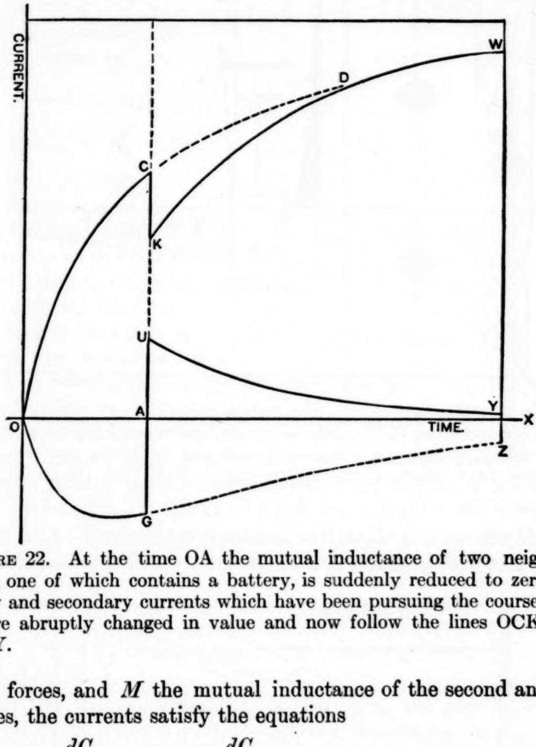

FIGURE 22. At the time OA the mutual inductance of two neighboring circuits, one of which contains a battery, is suddenly reduced to zero. The primary and secondary currents which have been pursuing the courses OCD, OGZ are abruptly changed in value and now follow the lines OCKW and OGAUY.

motive forces, and M the mutual inductance of the second and third branches, the currents satisfy the equations

$$(L + L_1)\frac{dC_1}{dt} + (L + M)\frac{dC_2}{dt} + (r + r_1)\,C_1 + rC_2 = E + E_1,$$

$$(M + L)\frac{dC_1}{dt} + (L + L_2)\frac{dC_2}{dt} + rC_1 + (r + r_2)\,C_2 = E + E_2, \tag{32}$$

$$C = C_1 + C_2.$$

Any sudden changes in the inductances cause such sudden changes in the currents as shall keep $[(L + L_1)\,C_1 + (L + M)\,C_2]$ and

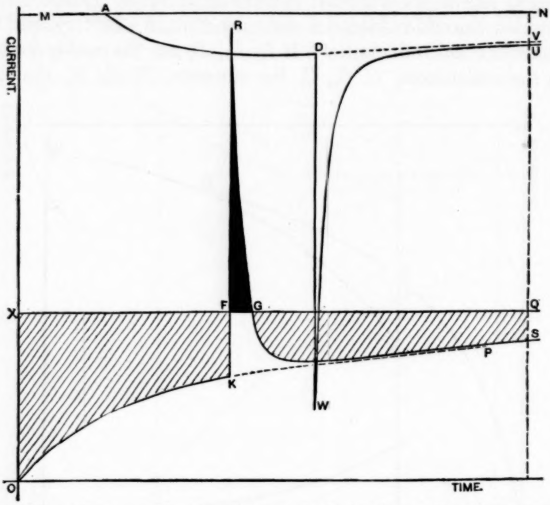

FIGURE 23. The two circuits of a certain induction coil without iron have equal self-inductances (L, L) and the mutual inductance $L\sqrt{2}$. At the time XF the mutual inductance is suddenly increased to $(1.9)L$, and the currents which have been following the curves OKP, ADV, take the forms KRGS, DWU.

$[(L + M)\,C_1 + (L + L_2)\,C_2]$ momentarily unchanged. In a case frequently met with in practice, there is no appreciable inductance in the

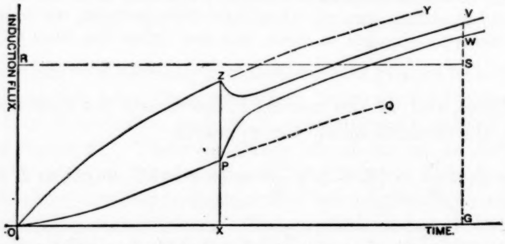

FIGURE 24. OPW and OZV show, as functions of the time, the fluxes of magnetic induction linked with the two circuits which carry the currents represented in the last figure.

first branch and no electromotive forces in the other branches, so that $L = 0$, $E_1 = 0$, $E_2 = 0$, and any instantaneous change in the inductances will leave

$$(L_1C_1 + MC_2)$$

and $(MC_1 + L_2C_2)$

momentarily unchanged.

If in Figure 14

$$P = L_1 - M,$$

$$Q = L + M,$$

$$R = L_2 - M,$$

and if the vanes are of such dimensions as to make the air resistance r_1, r, r_2 when the bodies to which they are attached have unit velocities, the equations of motion of the mechanical system are of the form (32), if E is applied upward to Q.

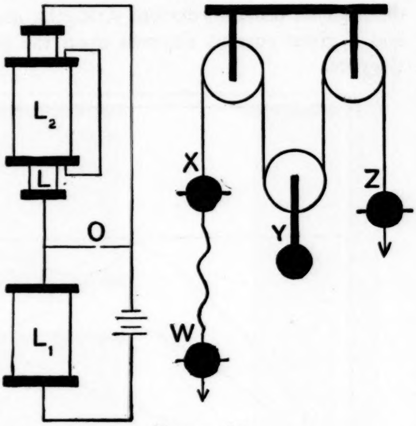

FIGURE 25.

Figure 26 illustrates a special case under this problem where $r = 1$, $r_1 = 20$, $r_2 = 30$, $L_1 = 2$, $L_2 = 3$, $M = 0$, up to the time OC, when by a sudden change in the conformation of the circuit, M is made equal to 2. Before the change $C_1 = 1.986$, $C_2 = 1.324$; the change in M leaves C_1 momentarily unchanged but suddenly reduces C_2 to zero. After the impulse the currents are given by the equations

$$\begin{aligned} C'_1 &= 3 - 1.717\, e^{-5.959t} + 0.703\, e^{-54.541t}, \\ C'_2 &= 2 - 1.428\, e^{-5.959t} - 0.572\, e^{-54.541t}, \end{aligned} \qquad (33)$$

very nearly.

If in the arrangement shown in Figure 9_e, the gap O, which has been closed by a stout wire, is suddenly opened, the current falls impulsively to a value which keeps the induction flux through the battery circuit momentarily unchanged.

The mechanical system shown in Figure 27 is analogous to the electrical circuit indicated in Figure 9_f. The gap O, which has been closed, is supposed to be opened at a given signal. The spring S is the analogue of the condenser K.

A circuit which contains an electromagnet has, of course, no definite inductance in the sense of the ratio of the flux of magnetic induction linked with the circuit to the intensity of the current, for this ratio is different for different current strengths, and for a given electromagnet, and a given current depends upon the previous magnetic history of the core.

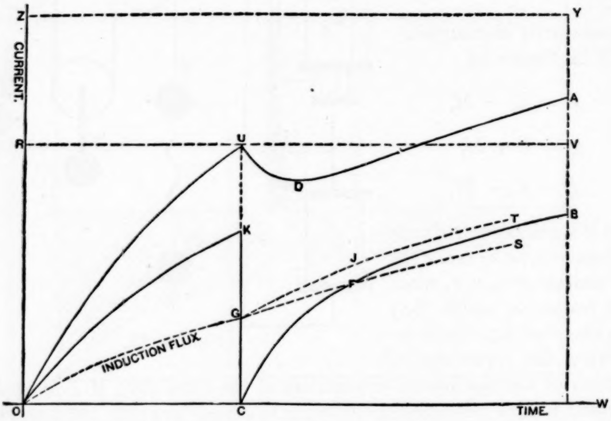

FIGURE 26. Two parallel coils of resistances 20 ohms and 30 ohms and of inductances 2 and 3 respectively, connect, in parallel, the terminals of a storage battery of 1 ohm internal resistance. At the beginning there is no mutual inductance between the branch circuits and the currents follow the curves OU, OK. At the time OC, the conformation of the circuit is suddenly changed so as to introduce a mutual inductance of 2, and as a result, the courses of the two currents are altered: the first follows henceforth the line UDA, the second the line KCFB. The inductions linked with the parallel branches are shown by the dotted curves.

In the case of a single circuit without iron the magnetic flux which accompanies a changing current is at every instant the same as it would be under a steady current of the intensity which the changing current then has. If, however, a second circuit closed on itself is brought into such a position that the two circuits have a mutual inductance, a changing current in the first circuit induces a current in the other which contributes to the flux through the first. If, therefore, an electromagnet has a solid core, the eddy currents induced in it while a current is growing or decreasing in the exciting coil affect the amount of the flux through the core, and it is not possible to obtain a hysteresis

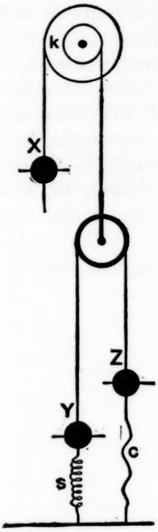

FIGURE 27.

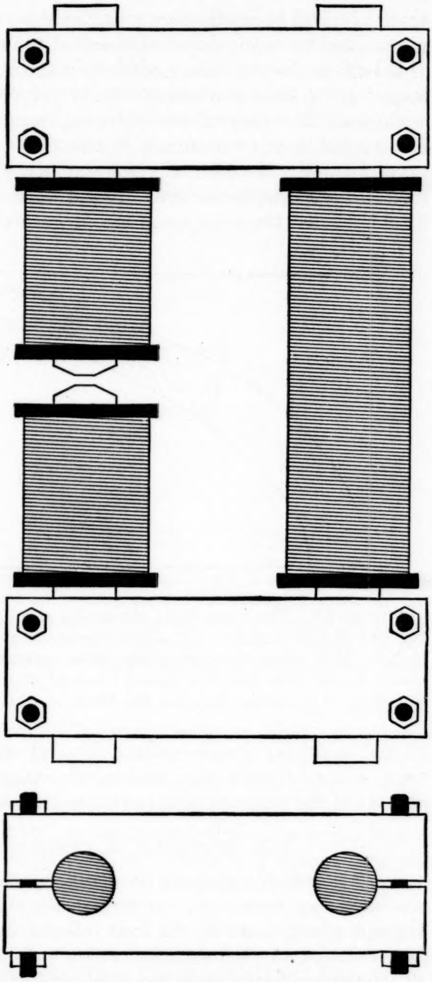

FIGURE 28. The electromagnet TP which has a solid core weighing about 300 kilograms.

diagram for the iron directly from the records of an oscillograph in the coil circuit. It is difficult, indeed, to obtain by any method a satisfactory magnetic curve for such a core, for if the iron starts from a given magnetic state, it is possible to get very different magnetic fluxes from a given exciting current by building up the current more or less slowly. In any useful examination of the magnetic properties of a solid piece of iron which is to be used for any practical purpose, it is essential

that the metal be made to go over the same magnetic journeys which it will later be required to make, and at the same speed.

Before we discuss this anomalous magnetization more carefully, we may stop for some moments to study the records of an oscillograph in circuits which contained either the electromagnet TP, Figure 28, which has a solid core, or a certain toroid (DN), about 41 centimeters in mean diameter, the core of which was made of about 25 kilograms of fine, soft, varnished iron wire. It will be seen from Figures 29, 30, 31, 32, 33, 34 that the phenomena are in general what we should expect

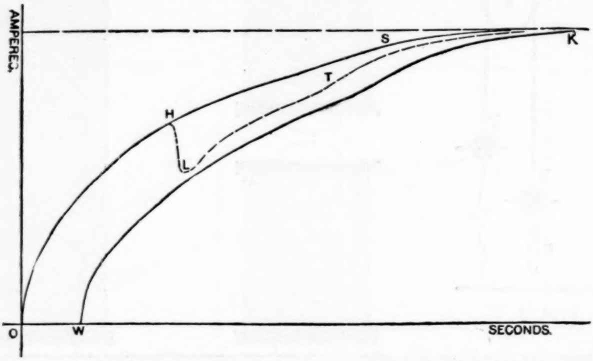

FIGURE 29. The curve OHS shows the manner of growth of a current in the coil of the magnet TP when the poles are separated by about three inches. WK shows the rise of the same current when the poles are nearly closed by the insertion of a planed block of iron between them. HLT shows the effect of suddenly dropping the block in while the current is growing.

to find in similar circuits without iron, though eddy currents and the time taken to make the mechanical changes modify somewhat the courses of the currents in the exciting circuits.

The Anomalous Magnetization of Iron.

In 1863 von Waltenhofen first called attention to the fact that if an increasing current (C) ending in the maximum value (C') be sent through a long solenoid, the final value of the magnetic moment of a bar of soft iron in the solenoid, which was at the outset demagnetized, will depend not only upon the final strength of the current, but also upon the manner of growth of C in attaining this intensity. This moment will be greater if the current be suddenly applied in full strength than if it be made to grow slowly, either continuously or by

short steps. If, after the current has remained steady for a short time at the strength C' it be made to decrease to zero, the residual moment of the bar will be less if the circuit be suddenly opened than if the decrease be made slowly by introducing more and more resistance.[4] If the soft iron bar to be magnetized was stout and relatively short, von Waltenhofen was sometimes able to reverse the direction of the remanent magnetism by a sudden break of the circuit. In one instance where the length of the bar was about ten centimeters and the diam-

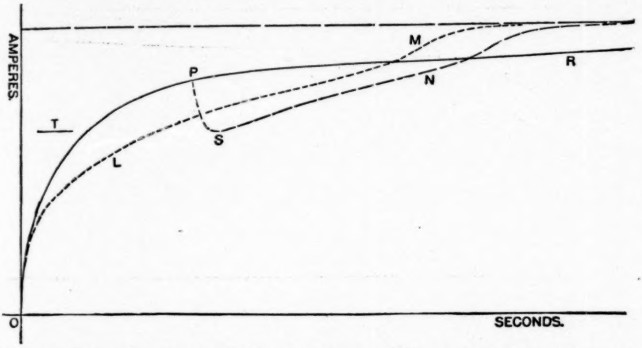

FIGURE 30. The short coils of the magnet TP are in series with one another and with a battery and an oscillograph. The current follows the line OPR or the line OLM according as the long coil of the magnet is open or closed. When at the point P, the long coil circuit is suddenly closed, the current in the battery follows the line PSN, which despite eddy currents is not very unlike the upper part of OLM.

eter about two centimeters, the magnetic moment while the current was passing was about 45 units, and about —0.20 when the current had been stopped. It seemed to von Waltenhofen that these phenomena could not be due to the induced currents caused by the sudden changes in the exciting current, and he explained them as consequences of the inertia of the molecular magnets turning in a viscous medium. This view seems to have been taken by Fromme, Auerbach, Ewing,

[4] Von Waltenhofen, Poggendorff's Ann. **120**, 1863; Fromme, Poggendorff's Ann., Ergbnd. **7**, 1876; Wied. Ann. **4**, 1878, **5**, 1878, **13**, 1881, **18**, 1883, **44**, 1891; Bartoli and Alessandri, Nuovo Cimento, **8**, 1880; Righi, Mem. di Bologna, **1**, 1880; Peuckert, Wied. Ann. **32**, 1887; Auerbach, Wied. Ann. **14**, 1881, **16**, 1882; Winkelmann's Handbuch der Physik, Band 5; Wiedemann, Lehre von der Elektricität, Band IV; Ewing, Magnetic Induction, § 84; Gumlich und Schmidt, Electrotechnische Zeitschrift, **21**, 1905.

Peuckert, Zielinski and others who have written upon the subject, while G. Wiedemann thought that his researches and those of Righi showed that eddy currents in the iron, and alternating currents induced in the exciting coil accounted best for the observed facts. In describing experiments upon this so-called anomalous magnetization, Wiedemann distinguishes between the permanent moment (P), that is, the remanent moment after the current has ceased, and the total moment (T), which

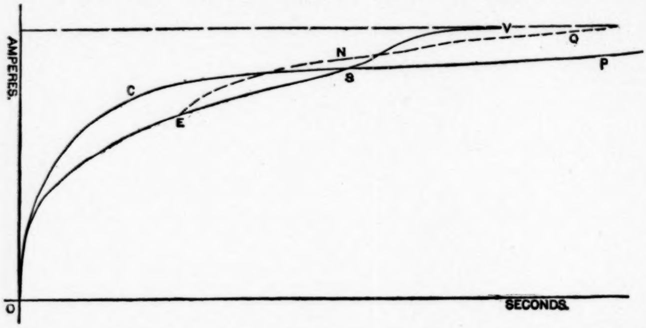

FIGURE 31. The short coils of the magnet TP are in series with one another and with a battery and an oscillograph. The current follows the line OCP or OESV according as the long coil of the magnet is open or closed on itself. When at the point E the long coil which has been closed is suddenly opened, the oscillogram is of the form ENQ which on account of the disturbing effects of eddy currents is not like the upper part of the curve OCSP.

is the moment when the current is steady at its highest value. This last quantity is regarded as the sum of the permanent moment and a moment (V) which vanishes with the current. The suffix a attached to P or T denotes that this moment has been reached after a gradual change in the current, while the suffix f denotes that the current has been suddenly opened or closed. According to Wiedemann, T_a is always smaller than T_f, and P_f than P_a, but these differences are much larger for short stout rods than for relatively long ones, where they become insignificant. $(P_a - P_f)/P_a$ is smaller in the case of a rod made of a bundle of insulated soft iron wires than in the case of a solid rod of the same dimensions. As C' is made larger, $T_f - T_a$ attains a maximum and then sometimes decreases slightly. If the rod to be magnetized is surrounded by a thick metal tube in which eddy currents can be induced, P_f is slightly increased, especially if the current be first slowly raised to C and then suddenly stopped. If shorter and

shorter iron rods of a given diameter are tested, P_f gradually decreases to zero under given value of C', and then changes sign; the inversion comes with longer rods when C' is weak than when it is strong. If, with a given rod, C' be gradually increased, the negative moment finally decreases and changes sign. After this there is no inversion and P_f is positive.

It appears from Fromme's experiments that the von Waltenhofen

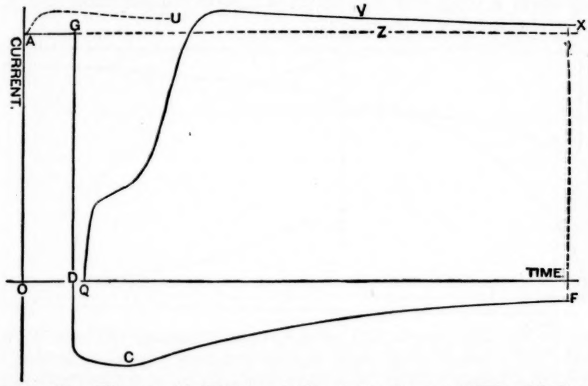

FIGURE 32. The toroid (DN) and the electromagnet (TP) with its poles separated by a gap of about eight centimeters, were placed in parallel across the terminals of a storage battery, with an oscillograph in the toroid branch. QVX shows the course of the current in the toroid which approaches its final value from above (See Figure 12). GCF, with its irregularities of curvature, shows the current in the toroid when the battery circuit was suddenly broken. AU shows the form of the temporary current induced in the toroid when an iron block was suddenly dropped into the gap between the jaws of the electromagnet, after the currents in the circuit had become steady.

effect is often less marked in straight, finely divided cores than in solid ones, and we may inquire how greatly the division of a straight core may be expected to facilitate the changes in the field (H) within the iron, due to given changes in the exciting circuit. It is clear that if the circuit of an electromagnet be suddenly broken, the decay of the electromagnetic field in the core is much less rapid when the core is solid and eddy currents induced in it shield the inner filaments, than when it is made of wire. Indeed, if eddy currents were non-existent, the field would fall instantaneously to zero, in the absence of magnetic lag, when the current in the coil ceased to flow. If the exciting coil remains closed and some change is suddenly made in its resistance or

in the electromotive force applied to it, the change in the current and therefore the change in the field in the iron caused by the current cannot be made instantaneous, even if eddy currents be wholly shut out, and, though dividing up the core has its effects, we cannot expect them to be so striking as in the case where the exciting circuit is open.

Let us consider a very long, uniformly wound solenoid consisting of N turns of insulated copper wire per centimeter of its length, wound closely upon a long, soft iron prism of square cross-section $(2a \times 2a)$

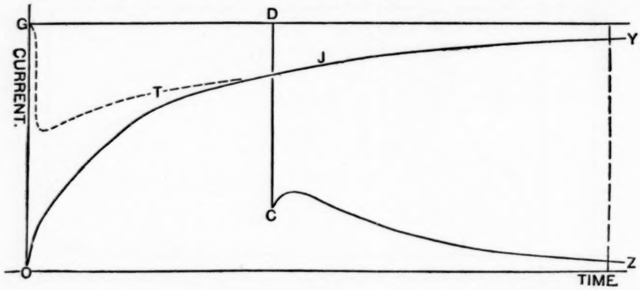

FIGURE 33. The toroid (DN) and the electromagnet (TP), with its jaws separated by a gap of about eight centimeters, were placed in parallel across the poles of a storage battery, with an oscillograph in the electromagnet branch. OJY shows the manner of growth of the current in TP and DCZ the manner of decay of the current when the battery circuit was suddenly broken. If, after the currents in the circuit have become steady, a block of iron was suddenly dropped into the gap in the core of the electromagnet, the induced current took the form GT.

built up compactly of a large number of straight, varnished filaments or "wires" of square cross-section $(c \times c)$, with their axes parallel to that of the prism, which shall be used as the z axis. The electric resistance of the solenoid per centimeter of its length parallel to the z axis shall be w, the constant applied electromotive force per centimeter of the axis shall be E, and the intensity of the current in the coil shall be C.[5] Within the core, the magnetic field (H) will have everywhere and always the direction of the axis of the prism, and if q is the current flux at any instant at any point in the iron, ρ the specific resistance of the metal, and μ its magnetic permeability, which for the present purpose shall be regarded as having a fixed uniform value, $q_x = 0,\ q_y = 0,\ H_x = 0,\ H_y = 0,\ H_z = H,\ 4\pi q = \text{Curl } H,$

[5] Peirce, These Proceedings, **43**, 5, 1907.

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y}\right). \tag{34}$$

When there are no eddy currents in the core, the intensity (H) of the magnetic field has at every point of the iron the boundary value $H_S = 4\,\pi NC$, but in general H varies from point to point. The flux of magnetic induction through the turns of the coil per centimeter of

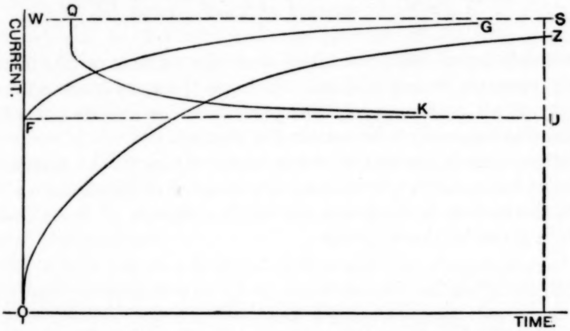

FIGURE 34. Two electromagnets were placed in series with each other and with an oscillograph and a storage battery, and a shunt (S) of small resistance was provided for one of the magnets. OZ shows the form of the battery current when S was closed, QK the fall of this current when S was suddenly opened after the original current had become steady, and FG the rise of the current to its old value when the shunt is again closed.

its length parallel to the z axis and N times the induction flux through the core are practically equal, and we may write

$$E - \frac{dp}{dt} = E - \mu N \int\int \frac{\partial H}{\partial t} \cdot dxdy = w \cdot C = \frac{w \cdot H_S}{4\,\pi N}, \tag{35}$$

or,

$$E = \frac{w \cdot H_S}{4\,\pi N} + \frac{\mu\rho N}{4\pi\mu}\int\int\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)dxdy, \tag{36}$$

where the integration extends over a cross-section of the core.

The vector H is always perpendicular to its curl, and the intensity of the component of the current at any point in the iron, in any direction s, parallel to the xy plane at any instant, is equal to $1/4\,\pi$ times the value at that point, at that instant, of the derivative of H in a direction parallel to the xy plane, and 90° in counter clockwise rotation ahead of s.

Along any curve in the iron parallel to the xy plane, H must be constant if there is no flow of electricity across the curve. At every instant, therefore, the value of H at the boundary common to any two filaments must be everywhere equal to H_S. If the coil circuit is broken, H must be constantly zero at the surface of every filament.

$$E = \frac{w \cdot H_S}{4\pi N} + \frac{\mu\rho n^2 N}{4\pi\mu}\int\int\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)dxdy, \tag{37}$$

where the integral is to be taken over the section of one filament.

Eddy currents in any filament which are the same in direction and intensity at all points on any line parallel to the axis do not affect in any way the magnetic field outside the filament.

If, after a steady current E/w has been running in the solenoid, the circuit be instantaneously broken, the value of H falls from $4\pi NE/w$ to 0 at the surface of the prism and at the surface of every filament, and H is given by the equation

$$H = \frac{16 H_0}{\pi^2}\sum_{j=0}^{j=\infty}\sum_{k=0}^{k=\infty}\frac{e^{-\lambda^2 t}}{(2j+1)(2k+1)}\cdot\sin\frac{(2k+1)\pi x}{c}\cdot\sin\frac{(2j+1)\pi y}{c}, \tag{38}$$

$$\lambda^2 = \frac{\pi\rho}{4\mu c^2}[(2k+1)^2 + (2j+1)^2], \tag{39}$$

in every filament. The origin is at one corner of the cross-section of the filament, and the x and y axes are two sides of the section; k and j are integers, and $H_0 = 4\pi NE/w$.

If we differentiate both members of equation (38) with respect to the time, and integrate the result over the cross-section of a filament, we get for the average value for the whole core, at the time t, of the rate of change of the magnetic force in the iron,

$$\frac{1}{c^2}\int\int\frac{\partial H}{\partial t}\cdot dS = \frac{4\rho H_0}{\mu\pi^3 c^2}\sum_{j=0}^{j=\infty}\sum_{k=0}^{k=\infty}\frac{[(2k+1)^2+(2j+1)^2]e^{-\lambda^2 t}}{(2j+1)^2(2k+1)^2} = \frac{4\rho H_0}{\mu\pi^3 c^2}\cdot M. \tag{40}$$

At the origin of time, M has the same maximum value for all values of c; and for different values of c (c', c''), M has the same numerical value at times (t', t'') such that $t'/c'^2 = t''/c''^2$, provided μ has the same value in both cases. The smaller the values of μ and c the sooner does M for a given core attain a given value. At $t = 0$, when the change of H in the iron is most rapid at all points, the average value

of dH/dt throughout the core is inversely proportional to c^2, that is, the average rate at which H is changing is 100 times as great when the core consists of filaments only one millimeter square as when the filaments are a centimeter square. This analysis illustrates the fact that when the main circuit of an electromagnet is suddenly broken, the changes in excitation to which the iron in a divided core is subject are far more violent than those which the particles of a solid core encounter. It is to be noticed that the average value of dH/dt given above is proportional to the electrical conductivity of the iron and to the intensity of the steady current in the exciting coil before the break.

In a similar manner, it is possible to show that if a given current in the exciting coil of an electromagnet be changed by a sudden increase or decrease in the resistance of the circuit while the applied electromotive force is unaltered, the whole given change of the magnetizing field in the core takes place somewhat more quickly if the core is finely divided than if it is solid. The general fact is, of course, evident without computation.

If the square core of a solenoid, the area of the cross-section of which is A square centimeters, be made of a bundle of infinitely long, straight iron wires placed close together, and if after a steady current of intensity E/w has been running for some time through the circuit so that there is a magnetic field of uniform intensity $H_0 = 4\,\pi NE/w$ in the core, the resistance of the solenoid circuit be suddenly changed to w' ohms per centimeter of length of the core, the current in the coil will gradually change to E/w', and the field in the core finally reaches the uniform value $H_\infty = 4\,\pi NE/w'$. At any instant the field in so much of the space A as is occupied by air is $4\,\pi NC$, for eddy currents in the round wires act like solenoidal current sheets, and do not affect the field outside the wires. Within each wire there are, of course, eddy currents, and at every point in the iron at every instant, the field intensity, H, must satisfy the equation (34).

The induction flux (p) through the solenoid per centimeter of its length is

$$4\,\pi N^2 C (A - n^2 B) + \mu N \int\int H \cdot dxdy, \tag{41}$$

where n^2 is the number of wires in the core and B is the area of the cross-section of each of them. The double integral is to be extended over the cross-sections of all the wires.

Since $$E - \frac{dp}{dt} = w'C,$$

$$w'C + (A - n^2B)\, 4\pi N^2 \cdot \frac{dC}{dt} + \mu N \int\!\!\int \frac{\partial H}{\partial t} \cdot dS = E, \tag{42}$$

and if H_S represents the strength of the magnetic field in the air space within the solenoid, and $A - n^2B$ is written $h \cdot A$,

$$H_S - \frac{4\,\pi NE}{w'} + \frac{4\,\pi N^2 hA}{w'} \cdot \frac{dH_S}{dt} + \frac{4\,\pi\mu n^2 N^2}{w'} \int\!\!\int \frac{\partial H}{\partial t} \cdot dS = 0, \tag{43}$$

where the double integral is to be taken over the cross-section of a single filament. If we put $H = H' + H_\infty$ and $H_S = H'_S + H_\infty$, the last equation becomes

$$H'_S + \frac{4\,\pi N^2 hA}{w'} \cdot \frac{dH'_S}{dt} + \frac{4\,\pi\mu n^2 N^2}{w'} \int\!\!\int \frac{\partial H'}{\partial t} \cdot dS, \tag{44}$$

in which H' satisfies at every point the equation

$$\frac{\partial H'}{\partial t} = \frac{\rho}{4\,\pi\mu}\left(\frac{\partial^2 H'}{\partial x^2} + \frac{\partial^2 H'}{\partial y^2}\right) = \frac{\rho}{4\,\mu\pi}\left\{\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial H'}{\partial r}\right)\right\} \tag{45}$$

where r is the distance from the axis of the wire in which the point lies. We are to find a function H' which satisfies equations (44), (45), which, when $t = 0$, is everywhere equal to $H_0 - H_\infty$, and which vanishes everywhere when t is infinite.

If
$$\varpi = \sum L e^{-\beta^2 t} \cdot J_0\,(mr), \tag{46}$$

in which either m or β may be chosen at pleasure and the other computed from the equation

$$m^2\rho = 4\,\pi\mu\beta^2, \tag{47}$$

and if for m in (46) we use the successive roots of the transcendental equation

$$J_0\,(mb)\left(1 - \frac{N^2 hA\rho \cdot m^2 b^2}{\mu w' b^2}\right) = \frac{2\,\pi n^2 N^2 \rho}{w'} \cdot mb \cdot J_1\,(mb), \tag{48}$$

where b is the radius of the wire, ϖ satisfies equations (44), (45) and vanishes when t is infinite.

Without any consideration of the question of a possible development of unity in terms of an infinite series of Bessel's Functions of the form $J_0\,(mr)$, where the m's have the values just mentioned, it is clear [6] that,

[6] Byerly, Annals of Mathematics for April, 1911.

within the comparatively short range from 0 to b, unity may be represented with sufficient accuracy by a few terms (sometimes two) of the form

$$L_1 \cdot J_0(m_1 r) + L_2 \cdot J_0(m_2 r) + L_3 \cdot J_0(m_3 r) + \ldots = \sum L_k \cdot J_0(m_k r), \quad (49)$$

so that
$$H = H_\infty + (H_0 - H_\infty) \sum L_k \cdot e^{-\beta^2 t} \cdot J_0(m_k r) \quad (50)$$

gives the value of the magnetic field at the time t at any desired point in the wire in question, and, therefore, at any desired point in any other wire of the core.

$$\frac{\partial H}{\partial t} = \frac{-(H_0 - H_\infty)\rho}{4\pi\mu} \sum L_k \cdot m^2 e^{-\beta^2 t} \cdot J_0(mr), \quad (51)$$

and if this be integrated over the cross-section of a wire and divided by πb^2, the result,

$$\Omega = \frac{-(H_0 - H_\infty)\rho}{2\pi\mu b^2} \sum L_k \cdot e^{-\beta^2 t} \cdot mb \cdot J_1(mb), \quad (52)$$

will represent the average value in the whole core, at the time t, of the time rate of change of the magnetic field. An example will best show the meaning of these rather intractable expressions.

Suppose the core of a long solenoid of square cross-section, ten centimeters on a side, to be built up of straight, round iron rods one millimeter in diameter placed close together; then $h = 0.2146$, $b = 0.05$, $n = 100$. If the resistance of the solenoid coil per centimeter of its length is $\frac{1}{16}$ of an ohm, the specific resistance of the iron 9950 absohms, the number of turns of wire per centimeter of the solenoid 10, and the value of the permeability of the iron 100, then $mb = x$ satisfies the equation

$$J_0(x) \cdot (1 - 1.3666\, x^2) = 1000\, x \cdot J_1(x), \quad (53)$$

and the first root $x = 0.04465$ will suffice, for $m = 0.8930$; and $J_0(0.8930\, r)$ differs from unity by less than one tenth of one per cent over the whole range from $r = 0$ to $r = b$, and from (50)

$$\frac{H - H_\infty}{H_0 - H_\infty} = e^{-6.315 t} \cdot J_0(mr), \quad (54)$$

very approximately. In the case of a core of the same cross-sectional area ($0.7854 A$), and the same permeability, but wholly without eddy currents, it is easy to show that

$$H = H_\infty + (H_0 - H_\infty) \cdot e^{-kt}, \quad (55)$$

where $k = w't/4\pi N^2 \cdot D$, $\quad D = A[4 + \pi(\mu - 1)]/4.$

For this problem, (55) yields

$$\frac{H - H_\infty}{H_0 - H_\infty} = e^{-6.316t}, \tag{56}$$

and a comparison of (54) and (56) shows that the eddy currents in a core of this wire, one millimeter in diameter, have practically no effect in slowing the changes in magnetism of the iron.

If the core of the given solenoid were made up of rods one centimeter in diameter, mb or x would be given as the roots of the equation

$$J_0(x) \cdot (1 - 0.01366\,x^2) = 10\,x \cdot J_1(x), \tag{57}$$

and it is not very difficult to show by a process of trial and error from Meissel's *Tafel der Bessel'schen Functionen*, that the first three of these roots are approximately equal to 0.4411, 3.8525, 7.0204, and that the corresponding values of $J_0(x)$ and $J_1(x)$ are 0.951946, $-$ 0.402672, 0.300112, and 0.215229, $-$0.008352, 0.001444.

If, with these roots, we wish to determine such a set of coefficients, (L_1, L_2, L_3) as shall make the mean square of the difference between unity and $\Sigma L \cdot J_0(mr)$ as small as possible, for the range from $r = 0$ to $r = b$, we have to solve the equations

$$A_1 \cdot L_1 + B_{12} \cdot L_2 + B_{13} \cdot L_3 = C_1, \quad B_{12} \cdot L_1 + A_2 \cdot L_2 + B_{23} \cdot L_3 = C_2$$

$$B_{13} \cdot L_1 + B_{23} L_2 + A_3 L_3 = C_3,$$ where

$$C_1 = 2\pi b^2 \cdot J_1(x_1)/x_1, \qquad A_1 = \pi b^2 \{[J_0(x_1)]^2 + [J_1(x_1)]^2\},$$

$$B_{12} = b^2\,[x_1 \cdot J_0(x_2) \cdot J_1(x_1) - x_2 \cdot J_0(x_1) \cdot J_1(x_2)]/(x_1^2 - x_2^2),$$

as Professor Byerly's theorems show. The computation here indicated shows that $\Omega = 6.096$ or 0.320, approximately, according as $t = 0$ or $t = 0.1$, whereas, if eddy currents were wholly cut out, the corresponding values would be 6.316 and 0.336. These figures illustrate the comparatively slight effect of subdividing the core in the particular case here considered. The results would, of course, be somewhat different numerically, with different assumed values for the constants of the circuit.

It is clear that the inversion of sign in the magnetic moment of a straight iron bar, when the magnetic excitation is suddenly removed, accompanies, at least, a large demagnetizing factor due to the ends of the bar, and no one seems to have observed the phenomenon in the case of closed cores. In rings, however, as in straight bars, the ulti-

mate value of the intensity (I) of magnetization depends very much upon the manner in which the given exciting current is made to attain its final strength.

The experiments of Rücker [7] upon small solid iron toroids seem to show that at moderate excitations there may be a difference of from 6 per cent to as much as 30 per cent in the final flux density due to a given current, according as the current is applied suddenly or by

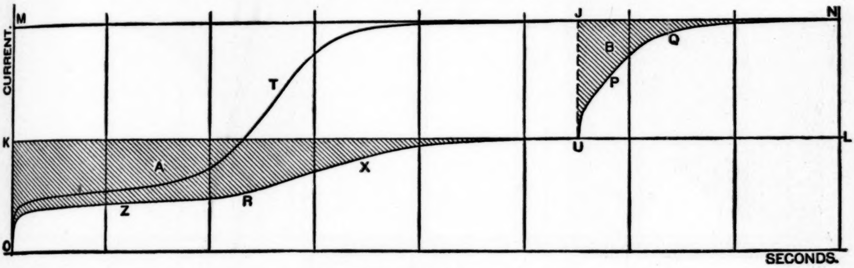

FIGURE 35.

many short steps, and, unlike some other observers, he found a very real difference ($T_f - T_a$), though a smaller one than in the case of the solid metal, for a toroid with core of fine iron wire (Blumendraht). In the case of a large electromagnet with solid, closed core, weighing altogether more than 1500 kilograms, Babbitt found, by a very ingenious method of procedure, a difference of 17.4 per cent between the final flux density in the iron caused by the sudden application of a given current, and the growth from nothing of the same current in 56 steps. The cross-section of this massive core is more than 450 square centimeters in its narrowest part, and eddy currents are so much in evidence that quite two minutes are required for a "suddenly applied" current to attain its steady value.

Babbitt also carried out a long series of very accurate measurements extended over several months, upon two small toroids of fine, carefully annealed iron wire, and upon a toroid weighing more than 40 kilograms made of very well softened iron wire about half a millimeter in diameter. His results show conclusively that if one of these softened and demagnetized cores has been first put through the cycle due to a given excitation a considerable number of times to obliterate the effects

[7] Babbitt, These Proceedings, **46**, 1911; Rücker, Inaugural Dissertation, Halle-Wittenberg, 1905.

of the past experiences of the iron, the form of the hysteresis diagram is precisely the same, whether the half cycle be carried out by one reversal of the main switch or in a very large number of steps. In general agreement with these results are some less accurate ones which I obtained three years ago in experimenting upon a transformer which has an exciting coil of 1394 turns and a core of about 120 square centimeters in cross-section, built up of thin strips of varnished sheet

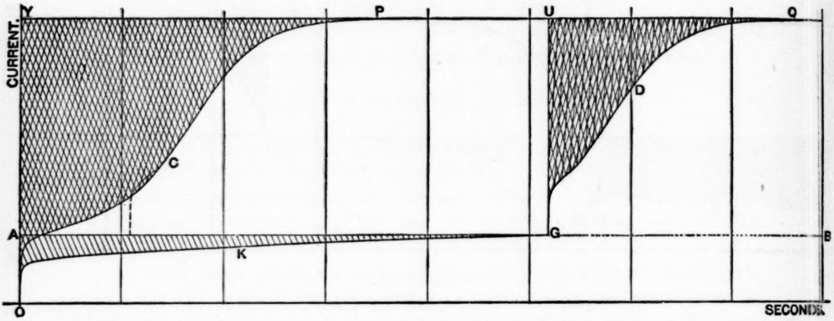

FIGURE 36. Growth from an originally neutral core of a current in a transformer with a laminated core. The effects of eddy currents are here noticeable.

iron about ten centimeters wide. This transformer was connected in simple circuit with a storage battery and a rheostat besides a suitable oscillograph. When the circuit was suddenly closed, with such a resistance (x) in the rheostat that the final strength of the current was about 1.10 amperes, the current curve was of the form R as shown in Figure 35, and when after a few seconds x was suddenly removed, so as to bring the final strength of the current up to about 2.30 amperes, the current curve was Q. When the whole journey was made without x the current curve was T. The sum of the flux changes represented by the shaded areas as measured by a Coradi "Grand Planimètre Roulant et à Sphère" was 1126, while the flux change corresponding to the area above the curve T was 1130. The core was not sufficiently well divided to avoid all evidence of eddy currents, for the curve Q does not exactly conform throughout with the upper part of T. This is shown more clearly in Figure 36, taken with the same transformer. Here the area of the shaded portion above K multiplied by the resistance then in the circuit should be equal to so much of the area above C, multiplied by the resistance belonging to it, as lies to the left of the

dotted line which rises at about 1.1 seconds after the circuit was closed, and is an exact copy of the curve D moved to the left. This curve coincides with C for a large part of its course, but has a trifle less area above it than that portion of C has which lies to the right of the ordinate at which the lowest part of the dotted curve begins. The shape of D just at the beginning points to the existence of eddy currents.

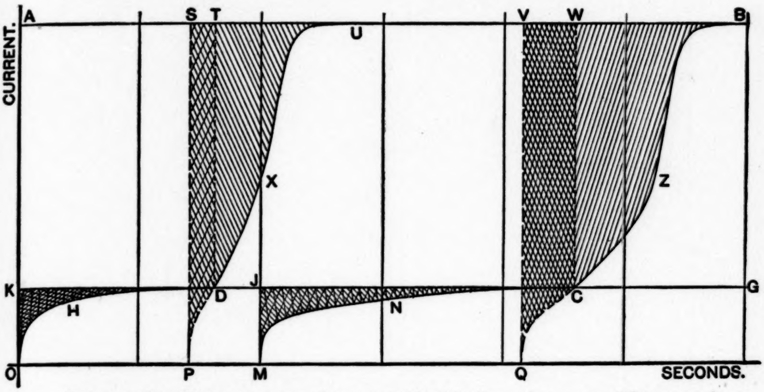

FIGURE 37. Current curves for a toroid with fine wire core. The second part of a two-stage current is exactly the same as if the current were allowed to grow at once to its final value.

To test more thoroughly the effect upon the flux of magnetic induction through the core of the transformer, of building up the current in different ways, I first measured with great care, by aid of a modified Rubens-du Bois "Panzer Galvanometer," the changes of this flux for a quick reversal of an excitation of 1812 ampere turns. I then measured by means of the planimeter a long series of oscillograph records obtained by reversing the same excitation by a considerable number of steps. All the testing instruments were different in the two cases, and no comparison was possible until the final results were reached and were found to differ from one another by only one part in fourteen hundred. The labor of reducing the oscillograms was so great that this close agreement must be considered accidental, but there can be little doubt, I think, that the flux change due to the single step and the sum of the changes due to the long series of steps which together cover the same change of excitation were practically indistinguishable.

Figure 37 shows copies of oscillograms taken with a number of toroids in series. The core of each toroid was made of perhaps fifteen

kilograms of very soft, varnished iron wire, about one tenth of a millimeter in diameter. The curves OHD, PDXU were taken when the

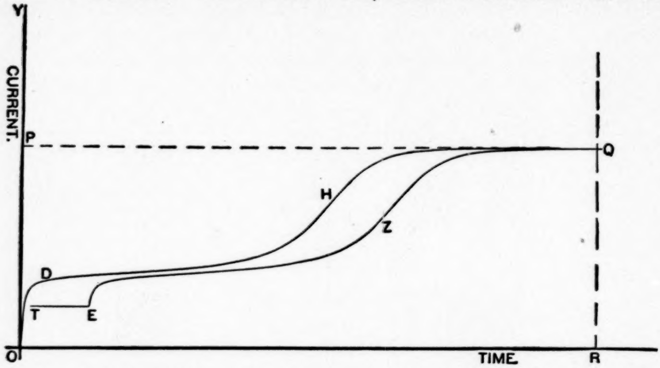

FIGURE 38. ODHQ represents the curve of growth of a current in the exciting coil of the toroid DN, if the circuit was suddenly closed when its resistance was r. If the circuit was first closed with a higher resistance $(r+s)$, which corresponded to a steady current of intensity OT, and if the resistance s was suddenly shunted out, the current rose to the intensity OP in the manner indicated by the curve EZQ, which, as Figure A shows, is of exactly the same form as the upper portion of ODHQ.

cores had been thoroughly demagnetized just before the experiment; the curves MNC, QCZB after the core had been put a number of times

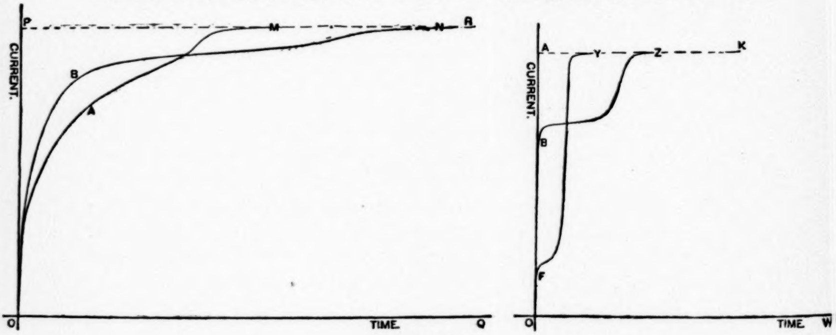

FIGURE 39.

through the cycle corresponding to the excitation used. The toroids were in simple circuit with a storage battery, an oscillograph, and a

rheostat of resistance x; when the circuit was suddenly closed the current grew to the final value corresponding to this resistance by the curve OHD or MNC, as the case might be. When at the proper time the rheostat resistance was suddenly shunted out of the circuit, the current rose to the value OA by the curve DXU or the curve CZB. If x had been shunted out at the start the current curve had the shape accurately represented, when the starting point had been shifted just

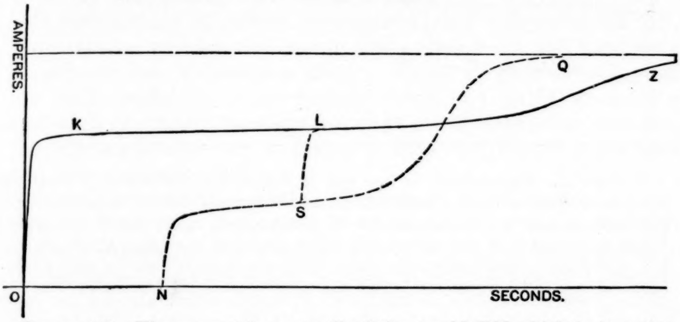

FIGURE 40. The current in one coil of the toroid DN which is in series with a battery follows the line OKLZ or the line NSQ, according as another coil on the same core is open or closed. When this last coil which has been closed is suddenly opened while the battery current is rising, this changes abruptly and follows exactly the upper part of the line OKLZ.

far enough to the right, by PDXU or QCZB. It was not possible to detect any difference between the curves DXU and CZB and the upper parts of the curves obtained with x all the time out of circuit. This figure was drawn by superposing several oscillograms, for it is very difficult after one curve has been taken upon the sensitized paper carried by the revolving drum to start another curve some time afterwards at such a point that it shall coincide with the upper part of the first one. This feat has, however, just been accomplished in another case by Mr. John Coulson, who made the records shown in Figures A and B, and has helped me in most of the experimental work of this paper. Figure 38, drawn from another photograph, shows the two curves which coincide in A. The oscillograph was in circuit with the coil of a large toroid of about 41 centimeters in diameter, the core of which is made of soft, varnished iron wire about half a millimeter in diameter. Each record shows a current curve obtained by applying the electromotive force directly to the circuit, and the second part of a current diagram when an extra resistance, at first in the circuit, was suddenly shunted out.

There seems to be in these cases neither a magnetic time lag nor any sensible von Waltenhofen effect.

If an electromagnet has two exciting coils, and if one of them be attached to the terminals of a battery, the form of the battery current

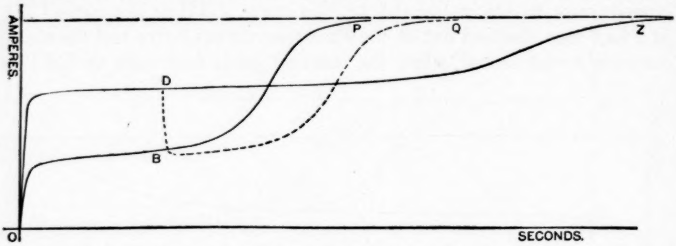

Figure 41. The current in one coil of the toroid DN which is in series with an oscillograph and a battery, follows the curve ODZ or the curve OBP, according as another coil wound on the same core is open or closed in itself. When at the point *D*, the second coil which has been open is suddenly closed, the oscillograph record gives the curve DBQ which except at the very beginning can be exactly superposed upon the upper part of the line OBP.

will depend upon whether the second coil is open or closed on itself, and the difference is usually noticeable even when the magnet has a

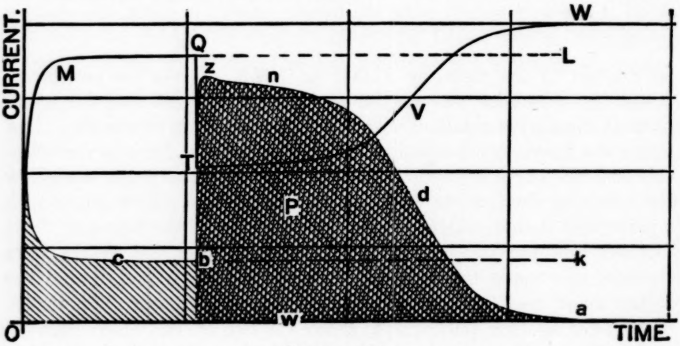

Figure 42.

large solid core in which eddy currents are being induced. Figure 39 shows curves taken under the two conditions just mentioned for both the electromagnet TP and the toroid DN. To determine whether the closing of the second coil in the case of the electromagnet where strong

eddy currents already existed changed the amount of the final flux through the circuit, Mr. Coulson has measured with great care a number of oscillograms taken with this apparatus, and finds the area between the asymptote and the curve OAM to be 6216 on the scale of his planimeter, while the area above the curve OBN is 6214 on the same scale. The areas above the curves agree within a small fraction of one per cent, as they were expected to do.

Figures 40 and 41 exhibit oscillograms taken with the toroid, DN, under sudden opening and closing of the second coil, and these show no signs of von Waltenhofen effects. Figure 42 gives the records of two oscillographs, one in the primary circuit of a toroid which has a core made of soft iron wire only one tenth of a millimeter in diameter, the other in a secondary coil, when a third coil, wound on the same core, was suddenly closed.

In early experiments upon the phenomenon of the reversal of moment in short rods magnetized in a solenoid, when the current was suddenly stopped, it was observed that if the rod had been previously magnetized permanently in the direction in which the current magnetized it, reversal never occurred, but that it always appeared, under favorable circumstances, if the direction of the previous magnetization was the opposite of that which the current gave it. This and like results has led many physicists to think that the molecules of the iron, when the exciting force due to the current is suddenly removed, return to the positions which they had just before the current acted upon them, but that the motion is so much resisted by frictional forces that the kinetic energy is lost when the particles have swung slightly beyond the positions of equilibrium where they are held by the friction. Wiedemann believed, on the other hand, that when the exciting circuit of an electromagnet is suddenly opened, the rise and decay of the Oeffnungsextrastrom induces in the mass of the iron, currents, alternating in direction and decreasing in intensity, and that the magnetization of a rod due to the original current is reversed in sign, under favorable circumstances, by a weaker current in the opposite direction. In the case of closed rings, where demagnetizing factors are absent, anomalous magnetization seems to appear only when eddy currents in the iron so shield the particles inside the mass that they are never exposed to sudden changes in the intensity of the exciting magnetic field.

My thanks are due to the Trustees of the Bache Fund of the National Academy of Sciences, who have lent me some of the apparatus used in measuring the oscillograms mentioned in this paper.

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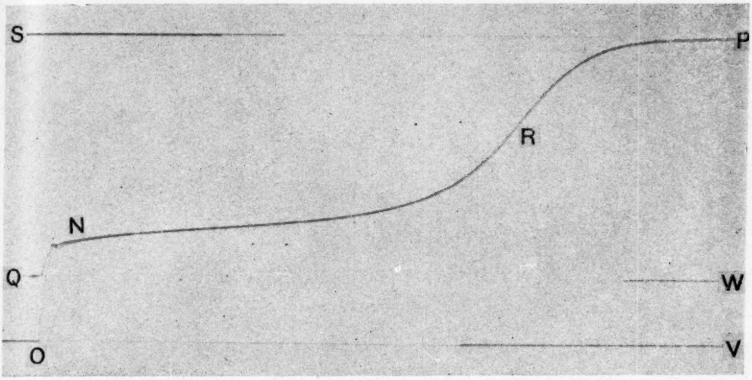
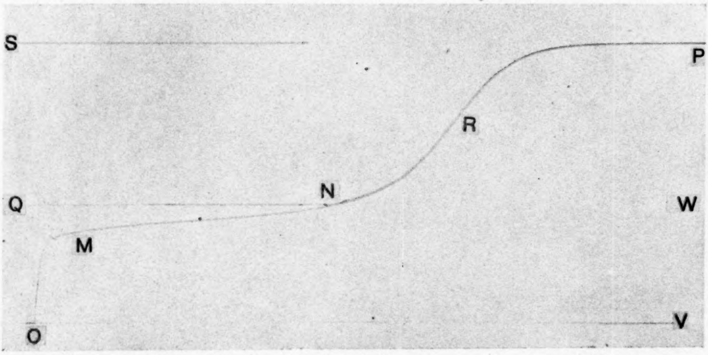


A.



B.